Riemannian Curvature and the Petrov Classification

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A connection is shown between the Riemannian Curvatures of the wave surfaces of a null vector l at a point p in space-time and the Petrov type of the space-time at p. Some other results on Riemannian Curvature are discussed.

1. Introduction

Let M be a Lorentzian space-time manifold. If $p \in M$, let $T_p(M)$ denote the tangent space to M at p and let $l \in T_p(M)$ be a null vector. Because of the identification of radiation in General Relativity with null geodesic congruences of curves in M, one is led to study the geometrical properties of the wave surfaces of l. These are the two dimensional subspaces of $T_p(M)$, each member of which is spacelike and orthogonal to l. There is in fact a two parameter family of such wave surfaces for a given null vector $l \in T_p(M)$ which, from the physical viewpoint, might be thought of as the totality of instantaneous wave surfaces of all observers with all possible velocities at p.* This two parameter family of wave surfaces can also be described as the orbit of one particular such surface under the action of a two parameter null rotation subgroup of the proper Lorentz group about the null vector l. These transformations are those null rotations about l for which l is the only fixed null direction and they constitute an abelian subgroup of the proper Lorentz group, being in fact isomorphic to the (translation) subgroup of the Möbius group which have only one fixed point (the point at infinity) on the extended complex plane. (See for example [2].)

This note will investigate the *Riemannian (sectional) curvature* of the wave surfaces of a given null vector l and under what conditions the Riemannian curvature is the same for all wave surfaces of l [c.f. 3, 4]. These conditions turn out to be associated with the Petrov type of the Weyl tensor

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* It should be pointed out that if the null vector l is part of a geodesic congruence of curves in M, then this congruence need not possess finite wave surfaces. This will occur when and only when the congruence is hypersurface orthogonal (twistfree) [1].

at the point p under consideration. (A full discussion of the Petrov classification and the geometry of null congruences and wave surfaces can be found in the works of Ehlers and Kundt [5], Sachs [6, 7], Pirani [8, 9] and Petrov [10]. The importance of the Petrov classification in General Relativity in the study of exact solutions of Einstein's equations is dealt with thoroughly in Ref. [5] and in a more recent paper by Kinnersley [11].)

2. Auxiliary Results

The notation will be the usual one. The Weyl tensor, Riemann tensor, Ricci tensor and Ricci scalar are, in local coordinates, connected by the component relation

$$C_{abcd} = R_{abcd} + R_{c[a}g_{b]d} + R_{d[b}g_{a]c} + \tfrac{1}{3} R\, g_{c[b}g_{a]d}. \tag{1}$$

Denoting the duality operator by an asterisk, one also has

$$^*R^*_{abcd} + R_{abcd} = 2g_{a[d}S_{c]b} + 2g_{b[c}S_{d]a}$$
$$(S_{ab} = R_{ab} - \tfrac{1}{4} R\, g_{ab}). \tag{2}$$

Equation (1) shows that $C_{abcd} = R_{abcd}$ if and only if $R_{ab} = 0$, whilst Eq. (2) yields the equivalences

$$^*R^*_{abcd} = -R_{abcd} \Leftrightarrow {}^*R_{abcd}$$
$$= R^*_{a\ bcd} \Leftrightarrow S_{ab} = 0 \Leftrightarrow R_{ab} \tag{3}$$
$$= \tfrac{1}{4} R g_{ab}.$$

The last condition in Eq. (3) is equivalent to M being an Einstein space.

If V is a *non-degenerate (non-null)* two dimensional subspace of $T_p(M)$, its *Riemannian curvature* $K(V)$ is given by

$$K(V) = \frac{R_{abcd}\,\xi^a\,\eta^b\,\xi^c\,\eta^d}{2g_{\,a[c}g_{d]b}\,\xi^a\,\eta^b\,\xi^c\,\eta^d}$$
$$= \frac{R_{abcd}\,\xi^{[a}\eta^{b]}\,\xi^{[c}\eta^{d]}}{(\xi^a\,\xi_a)(\eta^b\,\eta_b) - (\xi^a\,\eta_a)^2} \tag{4}$$

where ξ^a and η^a are the components of any basis for V, $K(V)$ being independent of the basis chosen.

Finally it is convenient to introduce at p a complex null tetrad of vectors $(l, m, t, \bar{t})$ where l and m are real null vectors, t and $\bar{t}$ are complex null vectors and the only non-zero inner products between the tetrad members are $l^a m_a = t^a \bar{t}_a = 1$. The complex vector t is assumed oriented so that the complex bivectors

$$V_{ab} = 2 l_{[a} \bar{t}_{b]},$$
$$M_{ab} = 2 l_{[a} m_{b]} + 2 \bar{t}_{[a} t_{b]} \qquad (5)$$
$$U_{ab} = 2 m_{[a} t_{b]}$$

are self dual;

$$V_{ab} = i \overset{*}{V}_{ab} \quad M_{ab} = i \overset{*}{M}_{ab} \quad U_{ab} = i \overset{*}{U}_{ab}.$$

The complex self dual Weyl tensor

$$\overset{+}{C}_{abcd} = C_{abcd} + i \overset{*}{C}_{abcd}$$

can now be expanded in terms of the bivectors (5) and five complex scalars C^i $(1 \leq i \leq 5)$ [6]

$$\overset{+}{C}_{abcd} = C^1 V_{ab} V_{cd} + C^2 (V_{ab} M_{cd} + M_{ab} V_{cd})$$
$$+ C^3 (U_{ab} V_{cd} + V_{ab} U_{cd} + M_{ab} M_{cd})$$
$$+ C^4 (U_{ab} M_{cd} + M_{ab} U_{cd})$$
$$+ C^5 U_{ab} U_{cd}. \qquad (6)$$

3. The Petrov Types

Let $l \in T_p(M)$ be a null vector and let $x, y \in T_p(M)$ be spacelike orthogonal unit vectors spanning a wave surface of l. Then any other wave surface of l can be obtained by performing one of the null rotations mentioned earlier on this wave surface. The transforms x' and y' of x and y respectively will span the new wave surface, where in components [2]

$$x'^a = x^a + \gamma l^a, \quad y'^a = y^a + \delta l^a. \qquad (7)$$

The two parameter family of wave surfaces of l is thus reflected in the two real parameters γ and δ.

Suppose now that the Riemannian curvatures of all wave surfaces of l at p are equal. By using (4) and (7) and the arbitrariness of γ and δ, one finds in terms of the vectors x and y that this is equivalent to

$$R_{abcd} F_1{}^{ab} F_2{}^{cd} = R_{abcd} F_2{}^{ab} F_2{}^{cd}$$
$$= R_{abcd} F_1{}^{ab} F_3{}^{cd}$$
$$= R_{abcd} F_3{}^{ab} F_3{}^{cd} \qquad (8)$$
$$= R_{abcd} F_2{}^{ab} F_3{}^{cd} = 0$$

where

$$F_1{}^{ab} = 2 x^{[a} y^{b]}, \quad F_2{}^{ab} = 2 l^{[a} x^{b]},$$
$$F_3{}^{ab} = 2 l^{[a} y^{b]} = - \overset{*}{F}_2{}^{ab}. \qquad (9)$$

The final equality in (9) gives the orientation of x and y. If one then defines the tensor

$$T_{ab} = R_{cadb} l^c l^d$$

at p, there results

$$T_{ab} x^a x^b = T_{ab} y^a y^b = T_{ab} x^a y^b = 0,$$
$$T_{ab} l^b = 0. \qquad (10)$$

There then follows the existence of real numbers μ, ν, λ such that

$$T_{ab} = \mu l_a l_b + \nu l_{(a} x_{b)} + \lambda l_{(a} y_{b)}. \qquad (11)$$

From the definition of T_{ab} this implies

$$l^b l^c l_{[e} R_{a]bc[d} l_{f]} = 0, \quad R_{ab} l^a l^b = 0. \qquad (12)$$

The Eqs. (12), (1) and (6) and the fact that l is null then imply that $l_{[e} \overset{+}{C}_{a]bc[d} l_{f]} = 0$, which means that either the Weyl tensor is zero or l is a Debever-Penrose vector at p [6].

Now suppose that l is an eigenvector of the Ricci tensor at p. From the symmetries of the Riemann tensor, one has

$$l^a R_{abcd} = l_b \overset{1}{H}_{cd} + x_b \overset{2}{H}_{cd} + y_b \overset{3}{H}_{cd}. \qquad (13)$$

Then the Eqs. (8) imply various algebraic relations on the $\overset{k}{H}_{ab}$ $(1 \leq k \leq 3)$. They imply in particular that the bivectors $\overset{2}{H}$ and $\overset{3}{H}$ are linear combinations of the bivectors $\overset{*}{F}_1, F_2$ and $\overset{*}{F}_2$. The statement that l is a Ricci eigenvector at p then shows that $\overset{1}{H}$ is a linear combination of $F_1, \overset{*}{F}_1, F_2$ and $\overset{*}{F}_2$. Finally, by contracting (13) with l^c and noting that $T_{[ab]} = 0$, one sees that the coefficient of $\overset{*}{F}_1$ in the expressions for $\overset{2}{H}$ and $\overset{3}{H}$ is zero. This gives $l^a l^c R_{abc[d} l_{e]} = 0$ which, because of (1) and (6) and since l is a null Ricci eigenvector, then yields $l^a l^c \overset{+}{C}_{abc[d} l_{e]} = 0$. So the Weyl tensor is either zero or else algebraically special at p with l as a repeated principal null direction.

Conversely suppose that the Weyl tensor is either zero or else algebraically special at p with l as a repeated principal null direction and that l is

a Ricci eigenvector at p. Then in (6), using a null tetrad based on l, one finds that $C^4 = C^5 = 0$ and so

$$\overset{+}{C}_{abcd} V^{ab} V^{cd} = \overset{+}{C}_{abcd} M^{ab} V^{cd} = 0. \tag{14}$$

If one converts from the null tetrad used in (14) to a real null tetrad under the identification

$$\sqrt{2} t_a = x_a + i y_a,$$

then because $*C_{abcd} = C^*_{abcd}$ and because l is a Ricci eigenvector, the Eqs. (8) for the Riemann tensor follow from (1). Hence the Riemann curvatures of all the wave surfaces of l are equal.

It is thus shown that for non conformally flat space-times:

(i) If the Riemannian curvatures of the wave surfaces of l are all equal, then l is a Debever-Penrose vector and (12) holds.

(ii) If l is a Ricci eigenvector then the Riemannian curvatures of the wave surfaces of l are all equal if and only if the Weyl tensor is algebraically special at p with a repeated principal null direction l.

If M is a vacuum, non flat space time, the conditions of (ii) automatically hold and a much simpler proof is available. One observes from (6) that the Riemannian curvature of the wave surface of l spanned by the complex vector t is $-\operatorname{Re}(C^3)$. On performing a null rotation (7) one evaluates the scalar C'^3 in the new tetrad. A simple computation then shows $-\operatorname{Re}(C^3)$ is invariant under null rotations if and only if $C^4 = C^5 = 0$. Alternatively one notes that *in vacuo*, (3) implies that the Riemannian curvatures of complimentary sub-spaces of $T_p(M)$ are equal. This can be used to extract further information from (8) which will imply equations like (14) for the Riemann tensor. The converse is similar. For vacuum space times, the value of the Riemannian curvature of the wave surfaces of l is zero for Petrov types N and III. It is in general non-zero for types II and D.

In the general case, it is remarked that the fact that there are at most four distinct Debever-Penrose directions and at most two distinct repeated principal null directions of the Weyl tensor at p, puts restrictions on the number of null directions whose wave surfaces all have the same Riemannian curvature. Finally, for any null Maxwell field, the principal null Maxwell direction l

has all its wave surfaces of the same Riemannian curvature since, in this case, $R_{ab} \propto l_a l_b$ and the Weyl tensor is either zero (whence one readily finds $l^a R_{abcd} = 0$ and so (8) holds) or else it is algebraically special with l as a repeated principal null direction [12].

4. The Riemannian Curvature of Orthonormal Tetrad Planes

On many occasions in the literature, restrictions on the Riemannian curvature of certain 2-spaces have been used in order to impose "symmetry conditions" on a manifold. As an example, it is recalled that if, say, M is a space-time manifold and if one asks that all 2-spaces at any given point $p \in M$ have the same Riemannian curvature α_p, then α_p is independent of p. This result, which is a consequence of the Bianchi identity, is the well known theorem of Schur (see for example [13]) and leads to the concept of a manifold of constant curvature. Another example has recently been discussed by Thorpe [4] who shows that for space-time manifolds, the function which associates a non-null 2-space with its Riemannian curvature takes its critical values on 2-spaces which determine what are effectively the non-null eigen-bivectors of the Riemann tensor. Thus for space-times which are Einstein spaces, Thorpe showed that the number of such critical points at an event in space-time determined the Petrov type of the Riemann tensor there. A third example might be briefly mentioned, namely Ricci's concept of mean curvature (based on Riemannian curvature) and Eisenhart's interpretation of it [13]. A fourth example is given in Reference [3].

In this section, it will be shown that in a general four dimensional Einstein space which has either positive definite or Lorentzian signature, one can (in infinitely many ways) always choose at each point an orthonormal tetrad which is "oriented symmetrically" in the sense that the Riemannian curvatures of the six coordinate planes are all equal. The proof in the positive definite case has already been given in a recent paper [14] but the proof was lengthy. Here a much simpler and more geometrical proof is given. In fact the proofs for the positive definite and Lorentzian signatures are almost identical and only one needs to be discussed.

Let M be a four dimensional Lorentzian manifold which is an Einstein space. If $p \in M$, let (x, y, z, t) be an orthonormal tetrad at p with

$$x^a x_a = y^a y_a = z^a z_a = -t^a t_a = 1 ,$$

let P_{xy} denote the two dimensional subspace of $T_p(M)$ spanned by x and y and so on for the other pairs of tetrad members and let K_{xy} denote the Riemannian curvature of P_{xy}. The following theorem can now be proved.

At any point p in a general 4 dimensional Einstein space which has either positive definite or Lorentzian signature one can always choose an orthonormal tetrad at p (in fact infinitely many) such that all the Riemannian curvatures K_{xy}, K_{yz}, K_{xz}, K_{xt}, K_{yt}, K_{zt} are equal.

To prove this, note that for coordinates about p such that $x^a = \delta_1{}^a$, $y^a = \delta_2{}^a$, $z^a = \delta_3{}^a$ and $t^a = \delta_4{}^a$ one has $g_{ab} = \mathrm{diag}(1, 1, 1, -1) = g^{ab}$ and so the Einstein space condition at p, $R_{ab} = \lambda g_{ab}$ $(\lambda \in \mathbb{R})$, gives for $a = b = 4$, $K_{14} + K_{24} + K_{34} = -\lambda$. Next, Eq. (3) shows that the Riemannian curvature of any 2-space is equal to that of its complement*. Hence $K_{12} + K_{23} + K_{31} = -\lambda$. To establish the above mentioned result, it is then sufficient to prove that a tetrad may be chosen at p for which the Riemannian curvatures of P_{12}, P_{23} and P_{31} are all equal. To do this one chooses an arbitrary tetrad at p, say (x, y, z, t) and performs a spatial orientation preserving rotation keeping t and z fixed

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* For four dimensional positive definite Einstein spaces the Riemann tensor satisfies *$R^*_{\mathrm{abcd}} = R_{\mathrm{abcd}}$ and so this result still holds.

so that in the new tetrad (x', y', z, t), $K_{x'z} = K_{y'z} = \alpha \in \mathbb{R}$. Then $K_{x'y'} = -(\lambda + 2\alpha)$. Now perform a similar rotation, keeping t and y' fixed, to a tetrad (x'', y', z'', t) where $K_{y'z''} = K_{y'x''}$. It then follows that $K_{x''z''} = \alpha$ and $K_{y'z''} = K_{y'x''} = -\frac{1}{2}(\lambda + \alpha)$. Now consider the function which associates with each tetrad (x, y, z, t) a real number according to the scheme

$$(x, y, z, t) \to (K_{xy} + \tfrac{1}{3}\lambda)(K_{xz} + \tfrac{1}{3}\lambda)(K_{yz} + \tfrac{1}{3}\lambda). \tag{15}$$

Here, only the restriction of this (continuous) function to the tetrads obtainable from the original one by spatial orientation preserving rotations will be considered. Since the associated subgroup of the Lorentz group is connected, the intermediate value theorem may be applied in the following way. If in the tetrad (x', y', z, t) $\alpha = -\frac{1}{3}\lambda$ then clearly

$$K_{x'y'} = K_{x'z} = K_{y'z} = -\tfrac{1}{3}\lambda$$

and the conclusion of the theorem holds. If however $\alpha + \frac{1}{3}\lambda \neq 0$, then the function (15) maps the tetrads (x', y', z, t) and (x'', y', z'', t) onto real numbers of opposite sign. One concludes the existence of a tetrad $(\bar{x}, \bar{y}, \bar{z}, t)$ in which one of the canonical 2-spaces, say $P_{\bar{x}\bar{y}}$, has $K_{\bar{x}\bar{y}} = -\frac{1}{3}\lambda$. Another rotation keeping $\bar{z}$ and t fixed yields a tetrad with the desired properties and completes the proof. The above argument also shows that infinitely many such tetrads at p exist because the first tetrad was arbitrary and only spatial transformations were used. The proof for the positive definite case is similar.

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